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LETTER TO THE EDITOR

The relativistic Bargmann transform

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Abstract. An integral transformation generalizing the Bargmann transform is established among two sets of wavefunctions associated with the configuration and Bargmann–Fock–Segal-like representations of a relativistic harmonic oscillator. The time-independent configuration-space wavefunctions are also studied and the lack of unitarity occurring when factorizing the time dependence of the wavefunction is solved by modifying the scalar product.

Recently [1], an $SO(1,2)$ algebra of operators $\hat{E}$, $\hat{x}$, $\hat{p}$, has been interpreted as the quantum symmetry algebra of a relativistic harmonic oscillator in $1+1$ dimensions, and realized in configuration space on a Hilbert space of wavefunctions generalizing the weight function (Gaussian) as well as the Hermite polymonials. Furthermore, an $SL(2,R)$ algebra of operators $\hat{E}$, $\hat{z}$, $\hat{z}^\dagger$, had been represented [2] in terms of creation and annihilation operators acting on complex analytic functions also generalizing the Bargmann–Fock–Segal (BFS) representation for the ordinary harmonic oscillator. It is then quite natural to look for the isomorphism between the two group representations that generalizes the Bargmann transform [3].

In this letter we wish to provide an integral transformation connecting the wavefunctions of the relativistic harmonic oscillator in configuration space and the corresponding Fock-like states. Let us specify the two sets we wish to relate. On the one hand, we have a Lie algebra of operators

$$[\hat{E},\hat{x}]=-\mathrm{i}\frac{\hbar}{m}\hat{p} \qquad [\hat{E},\hat{p}]=\mathrm{i}m\omega^2\hbar\hat{x} \qquad [\hat{x},\hat{p}]=\mathrm{i}\hbar\left(1+\frac{1}{mc^2}\hat{E}\right) \tag{1}$$

which is an affine version of the $SO(1,2)(1+1$ anti-de Sitter) algebra, giving the appropriate limits as $\omega\to 0$ and/or $c\to\infty$. The algebra (1) is realized by

$$\begin{aligned}
\hat{E}&=\mathrm{i}\hbar\frac{\partial}{\partial t}\\
\hat{p}&=-\mathrm{i}\hbar\alpha\cos\omega t\frac{\partial}{\partial x}+\mathrm{i}\hbar\frac{\omega x}{\alpha c^2}\sin\omega t\frac{\partial}{\partial t}+\frac{m\omega\sin\omega t}{\alpha}x\\
\hat{x}&=\frac{\cos\omega t}{\alpha}x+\mathrm{i}\hbar\frac{\alpha\sin\omega t}{m\omega}\frac{\partial}{\partial x}+\mathrm{i}\hbar\frac{x\cos\omega t}{\alpha mc^2}\frac{\partial}{\partial t}
\end{aligned} \tag{2}$$

acting on the configuration-space wavefunctions

$$\Psi_n^N(x,t) \equiv C_n^N \, e^{-in\omega t} \Phi_n^N(x) = C_n^N \, e^{-in\omega t} \alpha^{-(N+n)} H_n^N(\xi)$$

$$C_n^N = \sqrt{\frac{\omega}{2\pi}} \left(\frac{m\omega}{\hbar\pi}\right)^{1/4} \frac{1}{2^{n/2}\sqrt{n!}} \left(\frac{(2N-1)!(2N)^n}{(2N+n-1)!}\right)^{1/2} \left(\frac{\Gamma(N)}{\sqrt{N}\,\Gamma(N-\frac{1}{2})}\right)^{1/2} \tag{3}$$

where

$$\alpha \equiv \sqrt{1+\frac{\omega^2}{c^2}x^2} \qquad N \equiv \frac{mc^2}{\hbar\omega}$$

is a real number which measures how relativistic the system is

$$\xi \equiv \sqrt{\frac{m\omega}{\hbar}}\, x \qquad \text{and} \qquad H_n^N(\xi)$$

are polynomials generalizing the Hermite polynomials [1]. They are written as

$$H_n^N(\xi) = \sum_{s=0}^{[n/2]} \frac{(-1)^s n!}{s!(n-2s)!} \frac{N^s(N-\frac{1}{2})!}{(N+s-\frac{1}{2})!} \frac{(2N+n-1)!}{(2N)^n(2N-1)!} (2\xi)^{n-2s}. \tag{4}$$

Note in particular that the vacuum is characterized by the functions α^{-N} rather than the Gaussian one, which is in turn regained in the limit $N \to \infty$.

The scalar product defining the Hilbert space is as follows (C_n^N has been chosen in accordance with this scalar product):

$$\langle \Psi_n^N | \Psi_m^N \rangle = C_n^N C_m^N \int \mathrm{d}x \, \mathrm{d}t \, e^{-i(m-n)\omega t} \alpha^{-(2N+n+m)} H_n^N H_m^N = \delta_{nm} \tag{5}$$

where the integration measure is the invariant volume on the $SO(1,2)$ group once the p-dependence has been factorized out (unlike in the non-relativistic case, this requires a non-trivial regularization procedure [4]).

On the other hand, the affine version (trivial central extension) of the $SL(2,R) \approx SU(1,1)$ algebra

$$[\hat{h}, \hat{z}] = -2\hat{z} \qquad [\hat{h}, \hat{z}^\dagger] = 2\hat{z}^\dagger \qquad [\hat{z}, \hat{z}^\dagger] = \frac{1}{2N}\hat{h} + \hat{1} \tag{6}$$

generalizes the algebra of creation and annihilation operators of the non-relativistic harmonic oscillator in the sense that both $\omega \to 0$ and $c \to \infty$ limits are regained.

The isomorphism between the algebras (1) and (6) is established through the relations

$$\hat{h} = -\frac{2i}{\hbar\omega}\hat{E} \qquad \hat{z} = \frac{1}{\sqrt{2m\omega\hbar}}(m\omega\hat{x} + i\hat{p}) \qquad \hat{z}^\dagger = \frac{1}{\sqrt{2m\omega\hbar}}(m\omega\hat{x} - i\hat{p}). \tag{7}$$

Adopting normalized eigenstates of $\hat{E}$, $|n\rangle$, in the form

$$\langle 0|0\rangle = 1 \qquad |n\rangle = \frac{(\hat{z}^\dagger)^n|0\rangle}{\left(n!\prod_{s=1}^{n-1}\left(1+\frac{s}{2N}\right)\right)^{1/2}} \tag{8}$$

the quantum representation aquires the expression:

$$
\begin{aligned}
\hat{z}|n\rangle &= \left(n\left(1+\frac{n-1}{2N}\right)\right)^{1/2}|n-1\rangle \\
\hat{z}^{\dagger}|n\rangle &= \left((n+1)\left(1+\frac{n}{2N}\right)\right)^{1/2}|n+1\rangle \\
\hat{E}|n\rangle &= \hbar\omega n|n\rangle .
\end{aligned}
\tag{9}
$$

The formula (9) generalizes the non-relativistic Fock-space representation.

The algebra (6) is realized by

$$
\begin{aligned}
\hat{z}^{\dagger} &= \frac{e^{i\omega t}}{(1+\kappa)}\left(\frac{(1+\kappa)^2}{2}\frac{\partial}{\partial z}+\frac{z^{*2}}{N}\frac{\partial}{\partial z^*}-i\frac{a^*}{N\omega}\frac{\partial}{\partial t}-z^*\right) \\
\hat{z} &= -\frac{e^{i\omega t}}{(1+\kappa)}\left(\frac{(1+\kappa)^2}{2}\frac{\partial}{\partial z^*}+\frac{z^2}{N}\frac{\partial}{\partial z}+i\frac{z}{N\omega}\frac{\partial}{\partial t}+z\right) \\
\hat{h} &= \frac{2}{\omega}\frac{\partial}{\partial t}
\end{aligned}
\tag{10}
$$

when acting on complex functions

$$
\begin{aligned}
\tilde{\Psi}_n^N(z, z^*, t) &\equiv \tilde{C}_n^N\, e^{-in\omega t}\tilde{\Phi}_n^N(z, z^*) \equiv \tilde{C}_n^N\, e^{-in\omega t}\left(\frac{1+\kappa}{2}\right)^{-N-n} z^{*n} \\
\tilde{C}_n^N &= \sqrt{\frac{\omega}{2\pi}}\,\frac{1}{\sqrt{\pi n!}}\left(\frac{(2N+n-1)!}{(2N-1)!(2N)^n}\right)^{1/2}\left(\frac{2N-1}{2N}\right)^{1/2}
\end{aligned}
\tag{11}
$$

where $\kappa \equiv (1+(2zz^*/N))^{1/2}$. Note again that the vacuum $|0\rangle$ is associated with the function $((1+\kappa)/2)^{-N}$, which in the limit $N\to\infty$ leads to $e^{-zz^*/2}$. The scalar product in our BFS-like space is

$$
\langle\tilde{\Psi}_n^N|\tilde{\Psi}_m^N\rangle = \tilde{C}_n^N\tilde{C}_m^N\int\frac{d(\mathrm{Re}\, z^*)\, d(\mathrm{Im}\, z)}{\kappa}\, dt\, e^{-i(m-n)\omega t}\times\left(\frac{1+\kappa}{2}\right)^{-2N-n-m} z^n z^{*m} = \delta_{nm} \tag{12}
$$

where the integration measure is now the whole invariant volume on the $SL(2, R)$ group.

We now arrive at the central issue of this letter. The relativistic Bargmann transform (RBT) is intended to transform the energy eigenstates in the relativistic BFS space, i.e. $\langle z, t|n\rangle \equiv \tilde{\Psi}_n^N(z, z^*, t)$, into the energy eigenstates in configuration space, i.e. $\langle x, t|n\rangle \equiv \Psi_n^N(x, t)$, and vice versa. We formally write for the kernel $\langle x, t'|z, t\rangle$ of this integral transformation $\langle x, t'|n\rangle = \int (dz/\kappa)\, dt \langle x, t'|z, t\rangle\langle z, t|n\rangle$

$$
\begin{aligned}
\langle x, t'|z, t\rangle &= \sum_{n=0}^{\infty}\langle x, t'|n\rangle\langle n|z, t\rangle \\
&= \left(\frac{1+\kappa}{2}\right)^{-N}\alpha^{-N}\sum_{n=0}^{\infty} C_n^N\tilde{C}_n^N\alpha^{-n}H_n^N(\xi)\left(\frac{1+\kappa}{2}\right)^{-n} z^n e^{-in\omega(t'-t)}
\end{aligned}
\tag{13}
$$

which can be viewed as the generating function for the complete wavefunctions in configuration space.

We must now sum up expression (13). From Rodrigues' formula [5] for the polynomials $H_n^N(\xi)$, an integral representation for these can be found [6]

$$H_n^N(\xi)=(-1)^n\left(1+\frac{\xi^2}{N}\right)^{N+n}\frac{n!}{2\pi\mathrm{i}}\int_C\frac{\mathrm{d}s}{\left(1+\frac{s^2}{N}\right)^N(s-\xi)^{n+1}} \tag{14}$$

C being a contour including the point $s=\xi$ and exluding $\pm\mathrm{i}\sqrt{N}$. Putting this expression into (13) we obtain an integral representation for $\langle x, t'\,|\,z, t\rangle$:

$$\langle x, t'\,|\,z, t\rangle=\hat{C}^N\left(\frac{1+\kappa}{2}\right)^{-N}\alpha^{-N}\sum_{n=0}^{\infty}\frac{\alpha^{-n}}{2^{n/2}n!}\left(\frac{1+\kappa}{2}\right)^{-n}z^n\,\mathrm{e}^{-\mathrm{i}n\omega(t'-t)}$$
$$\times(-1)^n\left(1+\frac{\xi^2}{N}\right)^{n+N}\frac{n!}{2\pi\mathrm{i}}\int_C\frac{\mathrm{d}s}{\left(1+\frac{s^2}{N}\right)^N(s-\xi)^{n+1}} \tag{15}$$

$$\hat{C}^N=\frac{\omega}{2\pi}\frac{1}{\sqrt{\pi}}\left(\frac{m\omega}{\hbar\pi}\right)^{1/4}\left(\frac{2N-1}{2N}\right)^{1/2}\left(\frac{\Gamma(N)}{\sqrt{N}\,\Gamma(N-\frac{1}{2})}\right)^{1/2}.$$

In (15) the symbols $\int$ and $\sum$ can be interchanged and the geometrical series summed up. In fact

$$\sum_{n=0}^{\infty}(-1)^n z^n\mathrm{e}^{-\mathrm{i}n\omega(t'-t)}\left(\frac{1+\kappa}{2}\right)^{-n}\frac{\alpha^n}{2^{n/2}(s-\xi)^n}=\frac{(s-\xi)(1+\kappa)}{(s-\xi)(1+\kappa)+\sqrt{2}z\,\mathrm{e}^{-\mathrm{i}\omega(t'-t)}\alpha} \tag{16}$$

Since the integral

$$\int_C\frac{(1+\kappa)\,\mathrm{d}s}{\left(1+\frac{s^2}{N}\right)^N[(s-\xi)(1+k)+\sqrt{2}z\exp(-\mathrm{i}\omega(t'-t))\alpha]}$$

has a simple pole at

$$s_0=\xi-\frac{\sqrt{2}z\exp(-\mathrm{i}\omega(t'-t))\alpha}{1+\kappa}$$

the formula (15) can be finally computed:

$$\langle x, t'|z, t\rangle=\hat{C}^N\left(\frac{1+\kappa}{2}\right)^{-N}\alpha^N\left[1+\frac{s_0^2}{N}\right]^{-N}. \tag{17}$$

In the limit $N\to\infty$, $\langle x, t'\,|\,z, t\rangle$ leads to the kernel of the non-relativistic Bargmann transform

$$\langle x, t'|z, t\rangle^{\mathrm{NR}}=\frac{\omega}{2\pi}\frac{1}{\sqrt{\pi}}\left(\frac{m\omega}{\hbar\pi}\right)^{1/4}\mathrm{e}^{-|z|^2/2}\,\mathrm{e}^{\xi^2/2}\,\mathrm{e}^{-s_0^2} \tag{18}$$

$$=\frac{\omega}{2\pi}\frac{1}{\sqrt{\pi}}\left(\frac{m\omega}{\hbar\pi}\right)^{1/4}\mathrm{e}^{-z^2\exp(-2\mathrm{i}\omega(t'-t))/2+\sqrt{2}\xi z\exp(-\mathrm{i}\omega(t'-t))}\,\mathrm{e}^{-\frac{1}{2}(\xi^2+|z|^2)} \tag{19}$$

where

$$\tilde{s}_0 = \xi - \frac{1}{\sqrt{2}} z\, e^{-i\omega(t'-t)}.$$

The first exponential factor in (19) constitutes the generating function for the ordinary Hermite polynomials $H_n(\sqrt{(m\omega/\hbar)}x)$.

We must stress the different structure of the time evolution in the general theory of the relativistic harmonic oscillator as compared with the non-relativistic analogue. The time parameter cannot be factorized out in a natural way (see later). The appearance of the partial weights α^{-n} in the wavefunctions in configuration space or, $((1+\kappa)/2)^{-n}$ in BFS space (although the transformation

$$z \to \sqrt{\frac{2}{N}} \frac{z}{1+\kappa}$$

taking the complex plane to the open unit disk, hides the partial weights [7]), is traced back to the presence of a time derivative term in the quantum operators. Note that the RBT has to be established between the complete wavefunctions, i.e. including the partials weights, and, therefore, it is not possible to factorize the generating function of just the relativistic Hermite polynomials in the RBT (see [6] for the generating function of the polynomials H_n^N, i.e. without the weight function).

Another consequence of the structure of time evolution (manifest covariance of our configuration space representation) is the need for the time integration in the scalar product (5). In fact, a naive factorization of the time dependence in the wavefunctions, operators and scalar product leads to a non-unitary representation; the functions $\Phi_n^N(x)$ are no longer orthogonal nor the operators $\hat{z}$ and $\hat{z}^\dagger$ in (7) adjoint to each other. The x-representation can be nevertheless 'unitarized' by changing the integration measure, $dx \to dx/\alpha^2$, and by redefining the operators $\hat{z}$ and $\hat{z}^\dagger$ properly. After this changes the new scalar product and operators are

$$\langle \Phi_n'^N | \Phi_n'^N \rangle = C_n'^N C_m'^N \int \frac{dx}{\alpha^2} \Phi_n^{N*}(x) \Phi_m^N(x) = \delta_{nm}$$

$$C_n'^N = \sqrt{\frac{\omega}{2\pi}} \left(\frac{m\omega}{\hbar\pi}\right)^{1/4} \frac{1}{2^{n/2}\sqrt{n!}} \left(\frac{(2N-1)!(2N)^n}{(2N+n-1)!}\right)^{1/2} \left(\frac{N+n}{N}\right)^{1/2} \left(\frac{\Gamma(N+1)}{\sqrt{N}\,\Gamma(N+\frac{1}{2})}\right)^{1/2}$$

$$\hat{z}\Phi_n'^N = \frac{1}{\sqrt{2m\omega\hbar}} \left(\frac{N+n-1}{N+n}\right) \left[\hbar\alpha \frac{d}{dx} + \frac{N+n}{N\alpha} m\omega x\right] \Phi_n'^N \qquad (20)$$

$$\hat{z}^\dagger \Phi_n'^N = \left(\frac{1}{\sqrt{2m\omega\hbar}}\right)^{1/2} \frac{N+n+1}{N+n} \left[-\hbar\alpha \frac{d}{dx} + \frac{N+n}{N\alpha} m\omega x\right] \Phi_n'^N$$

with a new normalization factor $C_n'^N$. it should be noted that the operators $\hat{z}$ and $\hat{z}^\dagger$ have acquired an extra factor depending on n in addition to the naive replacement $i\hbar\partial/\partial t \to \hbar\omega n$.

The redefinition process really parallels the *multipliers method* used in the literature [7–9] to construct unitary representations of a group G when a natural invariant volume is absent. The measure dx/α^2 is left invariant under the U(1) subgroup of SO(1, 2), i.e. the time evolution, so that the energy operator is not affected by the multipliers.

Although the representation (20) is not manifestly covariant, it really constitutes a well-defined theory of orthogonal functions (polynomials with partial weights) tied to the relativistic harmonic oscillator.

Even though the BFS-like representation (9, 11) can be directly restricted to de z, z^*-dependence without losing unitarity, a time-independent RBT is not directly defined since the new family of representations (20) admits a larger spectrum of N. In fact, the allowed values of N are those for which the vacuum admits a finite norm, and C_n^N in (3) is defined for $N>\frac{1}{2}$ whereas $C_n'^N$ in (20) is finite also for $N>0$. It should be stressed that only half-integer values greater than $\frac{1}{2}$ correspond to single-valued representations of $SL(2, R)$ (the rest are associated with the universal covering group).

In order to construct a time-independent RBT we should first extend the BFS-like representation to admit these new values for N, and this means modifying the scalar product and operators in a way similar to that of (20). This is achieved by the following expressions:

$$\langle\tilde{\Phi}_n'^N|\tilde{\Phi}_n'^N\rangle = \tilde{C}_n'^N\tilde{C}_m'^N\int\frac{\mathrm{d}(\mathrm{Re}\, z^*)\,\mathrm{d}(\mathrm{Im}\, z)}{\kappa(1+\kappa)}\tilde{\Phi}_n^{N*}(x)\tilde{\Phi}_m^N(x)=\delta_{nm}$$

$$\tilde{C}_n'^N=\sqrt{\frac{\omega}{2\pi}}\frac{1}{\sqrt{\pi n!}}\sqrt{\frac{2(2N+n)!}{2(N)!(2N)^n}}$$

$$\hat{z}\tilde{\Phi}_n'^N=\left(\frac{2N+n-1}{2N+n}\right)^{1/2}\left(\frac{(1+\kappa)}{2}\frac{\partial}{\partial z^*}+\frac{z^2}{N(1+\kappa)}\frac{\partial}{\partial z}+\frac{z(N+n)}{(1+\kappa)N}\right)\tilde{\Phi}_n'^N$$

$$\hat{z}^\dagger\tilde{\Phi}_n'^N=\left(\frac{2N+n+1}{2N+n}\right)^{1/2}\left(-\frac{(1+\kappa)}{2}\frac{\partial}{\partial z}-\frac{z^{*2}}{N(1+\kappa)}\frac{\partial}{\partial z^*}+\frac{z^*(N+n)}{(1+\kappa)N}\right)\Phi_n'^N \qquad (21)$$

although other realizations are also allowed. The extended RBT now reads

$$\langle x|z\rangle=\hat{C}'^N\left(\frac{1+\kappa}{2}\right)^{-N}\alpha^N\frac{1}{2\pi i}\int_C\frac{\mathrm{d}s}{(1+(s^2/N))^N(s-\xi)}\sum_{n=0}^{\infty}(-1)^n$$

$$\times\left(\frac{(2N+n)(N+n)}{N}\right)^{1/2}\frac{\alpha^n}{2^{n/2}}\left(\frac{1+\kappa}{2}\right)^{-n}\frac{z^n}{(s-\xi)^n}$$

$$\hat{C}'^N=\frac{\omega}{2\pi}\frac{1}{\sqrt{\pi N}}\left(\frac{m\omega}{\hbar\pi}\right)^{1/4}\left(\frac{\Gamma(N+1)}{\sqrt{N}\,\Gamma(N+\frac{1}{2})}\right)^{1/2} \qquad (22)$$

where the series is convergent (by standard theorems on alternating series) even though it is no longer a geometric series. A more detailed study about canonical minimal representations (versus the manifestly covariant ones (2), (3), (5) and (10), (11), (12)) of the relativisitic harmonic oscillator, including mean creation and annihilation operators and their corresponding coherent states, will be given elsewhere.

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